Announcements

1) No more Webwork!

Which matrix goes first?

Start with a matrix A.

If A is diagonalizable,

then there exists a

diagonal matrix D

and an invertible

matrix S with

$$\boxed{S^{-1}AS = D}$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A and $v_1, v_2, \cdots, v_n$ the associated eigenvectors.

Then A is "diagonal" with respect to $\{v_1, v_2, \cdots, v_n\}$ in the sense that

$$\boxed{A v_i = \lambda_i v_i}$$

for $1 \le i \le n$.

Let $S = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$.

How do we know this is S and not S^{-1}?

We know

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$\boxed{De_i = \lambda_i e_i}.$$

$$S e_i$$

$$= \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} e_i$$

$$= i^{th} \text{ column of } S$$

$$= v_i$$

$$ASe_i$$

$$= A(Se_i)$$

$$= Av_i$$

$$= \lambda_i v_i$$

$$S^{-1}ASe_i$$

$$= S^{-1}(ASe_i)$$

$$= S^{-1}(\lambda_i v_i)$$

$$= \lambda_i S^{-1} v_i = \lambda_i e_i$$

Picture

$$\begin{tikzcd} e_i \arrow[r, "S"] \arrow[rrr, bend right, "= S^{-1}AS"'] & v_i \arrow[r, "A"] & \lambda_i v_i \arrow[r, "S^{-1}"] & \lambda_i e_i \end{tikzcd}$$

Example 1: Diagonalize

$$A = \begin{bmatrix} -2 & 56 \\ 56 & 11 \end{bmatrix}$$

$$\lambda_1 = \frac{1}{2}\left(9 + \sqrt{12713}\right)$$

$$\lambda_2 = \frac{1}{2}\left(9 - \sqrt{12713}\right)$$

$$v_1 = \begin{bmatrix} \frac{1}{112}\left(-13 + \sqrt{12713}\right) \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} \frac{1}{112}\left(-13 - \sqrt{12713}\right) \\ 1 \end{bmatrix}$$

In this case

$$S = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{112}\left(-13+\sqrt{12713}\right) & \frac{1}{112}\left(-13-\sqrt{12713}\right) \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \frac{1}{2}\left(9+\sqrt{12713}\right) & 0 \\ 0 & \frac{1}{2}\left(9-\sqrt{12713}\right) \end{bmatrix}$$

$$\boxed{S^{-1}AS = D}$$

Eigenvalues of Special Matrices

Let A be $n \times n$.

1) If $(Av) \cdot v \geq 0$,

then all eigenvalues of A are non-negative.

We call such an A

positive semi-definite.

2) If $A^t A = I_n$, then all eigenvalues of A are either 1 or -1. We call such a matrix **orthogonal**. We call A this because the condition implies the rows of A are all orthogonal to eachother and the columns of A are orthogonal to eachother.

Diagonalization

When can we diagonalize an $n \times n$ matrix A?

Precisely when A has n linearly independent eigenvectors! If $v_1, v_2, \ldots, v_n$ are these eigenvectors corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, then

$$S^{-1}AS = D$$

where $S = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}.$$

The symmetric case

Definition: An $n \times n$ matrix A is called symmetric if $\boxed{A = A^t}$.

Existence of Eigenvalues for Symmetric Matrices

Let A be 2×2, $A = A^t$.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

$$\det\left(A - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\begin{pmatrix} a-\lambda & b \\ b & c-\lambda \end{pmatrix}$$

$$= \lambda^2 - (a+c)\lambda + ac - b^2 = 0$$

$$\lambda^2-(a+c)\lambda+ac-b^2=0$$

Using quadratic formula,

$$\lambda=\frac{(a+c)\pm\sqrt{(a+c)^2-4ac+4b^2}}{2}$$

Can this be imaginary?

$$(a+c)^2-4ac+4b^2$$

$$=a^2+2ac+c^2-4ac+4b^2$$

$$=a^2-2ac+c^2+4b^2$$

$$= (a-c)^2 + 4b^2$$

$$\geq 0 \quad \text{always!}$$

So A only has real eigenvalues!

To go to the

$n \times n$ symmetric

matrices, we use

mathematical induction:

Use $n=1$ to prove $n=2$ then

Use $n=2$ to prove $n=3$ then

Use $n=3$ to prove $n=4$ then

Continue!

Orthogonal Diagonalization

If A is symmetric,

then $S^{-1}AS = D$

for S satisfying

$S^t S = I_n$ (orthogonal)

Then $S^{-1} = S^t$, so

$$\boxed{S^t A S = D}$$

To get this, let's show that if $\lambda_1 \neq \lambda_2$ are two eigenvalues of A where $A = A^t$, then if $A v_1 = \lambda_1 v_1$ and $A v_2 = \lambda_2 v_2$, we must have $v_1 \cdot v_2 = 0$.

Fact: I have new pen colors.

But the relevant fact is that for all v, w in $\mathbb{R}^n$,

$$\boxed{(Av)\cdot w = v\cdot(A^t w)}.$$

Then if $\lambda_1 = 0$, we have $\lambda_2 \neq 0$, and

$$v_1 \cdot v_2 = v_1 \cdot \left(\frac{\lambda_2}{\lambda_2} v_2 \right)$$

$$= \frac{1}{\lambda_2} v_1 \cdot (\lambda_2 v_2)$$

$$= \frac{1}{\lambda_2} v_1 \cdot (A v_2)$$

$$= \frac{1}{\lambda_2} v_1 \cdot (A^t v_2)$$

(since $A = A^t$)

$$= \frac{1}{\lambda_2} (A v_1) \cdot v_2$$

$$= \frac{1}{\lambda_2} \vec{0} \cdot v_2 = 0$$

If $\lambda_1 \neq 0$, then

$$
\begin{aligned}
v_1 \cdot v_2 &= \left(\frac{\lambda_1}{\lambda_1} v_1\right) \cdot v_2 \\
&= \frac{1}{\lambda_1} (\lambda_1 v_1) \cdot v_2 \\
&= \frac{1}{\lambda_1} (A v_1) \cdot v_2 \\
&= \frac{1}{\lambda_1} v_1 \cdot (A^t v_2) \\
&= \frac{1}{\lambda_1} v_1 \cdot (A v_2) \quad (A = A^t) \\
&= \frac{1}{\lambda_1} v_1 \cdot (\lambda_2 v_2) \\
&= \frac{\lambda_2}{\lambda_1} (v_1 \cdot v_2)
\end{aligned}
$$

Then subtracting

$$\frac{\lambda_2}{\lambda_1}(v_1 \cdot v_2),$$

$$v_1 \cdot v_2 - \left(\frac{\lambda_2}{\lambda_1}\right) v_1 \cdot v_2 = 0$$

so $(v_1 \cdot v_2)\left(1 - \frac{\lambda_2}{\lambda_1}\right) = 0.$

This means either $v_1 \cdot v_2 = 0$ or $1 - \frac{\lambda_2}{\lambda_1} = 0$. But we know $\lambda_2 \neq \lambda_1$, so $1 - \frac{\lambda_2}{\lambda_1} \neq 0$, and then we must have $v_1 \cdot v_2 = 0$ ✓

Problem: What if $\lambda_1 = \lambda_2 = \lambda$?

Then the subspace of all v in $\mathbb{R}^n$ with $Av = \lambda v$ is at least two dimensional, so we can choose v_1 and v_2 with $v_1 \cdot v_2 = 0$ in this subspace.

We can then get

$v_i \cdot v_j = 0$ for all $i \neq j$.

By dividing by $\|v_i\|_2$, we may assume $\|v_i\|_2 = 1$, and so $S = [v_1 \ v_2 \cdots v_n]$ satisfies $S^t S = I_n$.